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The metric classification of conics and quadrics

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THE METRIC CLASSIFICATION

OF

CONICS AND QUADRICS

BY

GEORGE RUTLEDGE

THESIS

FOR THE

DEGREE OF BACHELOR OF ARTS

IN

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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

George Rutledge

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IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE

DEGREE OF Bachelor of Arts

in General Science

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PREFACE

This thesis is designed to present, in a form suitable for use in classes of Sophomores or Juniors, a purely analytic treatment of the general problem of the metric classification of conics and quadrics, leading up to the simple rule for determining the coefficients of the reduced equation, which is given by Bromwich in his book on "Quadratic Forms and their Classification by means of Invariant Factors."

The plan was suggested by a course of lectures on solid geometry, given by Professor Chas. N. Haskins at the University of Illinois in 1908-09, and much of the material is due to him.

Theorem 3 of Chapter I. is fundamental to the whole treatment, and the demonstration here given is an obvious improvement over that given by Bromwich. The results stated in Chapter IV. are believed to be new and will appear in a forthcoming number of the Annals of Mathematics.

For the reading of the thesis, such knowledge of determinants and linear equations as is contained in the first five Chapters of Bôcher's Introduction to Higher Algebra is presupposed.

Chapter I. is designed to furnish simple proofs of such theorems from the theory of determinants as are essential to the reading of the chapters following, but which are not ordinarily familiar to the student who has had merely an introductory course in determinants.

George Rutledge.

University of Illinois.

May, 1910.

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CONTENTS

Page

CHAPTER I. INTRODUCTORY

Art. 1. Purpose of the introductory chapter. . . . 1

2. Effect of a linear substitution on a quadratic form and on its determinant. 1

3. Product of determinants. 4

4. Reality of the roots of the secular equation. . 7

CHAPTER II. REDUCTION OF THE GENERAL EQUATION.

5. Effect of an orthogonal substitution. . . 10

6. Determination of the reducing substitution when the roots of the reducing cubic are distinct. . 11

7. Existence of the reducing substitution when the reducing cubic has multiple roots. . . . 13

8. Determination of the reducing substitution when the reducing cubic has multiple roots. . . 17

9. Elementary divisors. 19

10. Final simplification. 21

CHAPTER III. CLASSIFICATION OF THE REDUCED FORMS.

11. Direct determination of the coefficients of the reduced equation from those of the original. . 23

12. Examples on the method of the text. . . . 26

CHAPTER IV. CLASSIFICATION BY MEANS OF RANK

13. Direct determination of type. 27

14. Example. 29

CHAPTER I.

INTRODUCTORY.

1. It is the purpose of this introductory chapter to present formal proofs of certain well-known theorems which will find frequent application in the following pages.

2. EFFECT OF A LINEAR SUBSTITUTION ON A QUADRATIC FORM AND ON ITS DETERMINANT.

Theorem 1. Let the homogeneous quadratic form, $f(x_1, x_2, \ldots, x_n) \equiv \sum_{i}^{n} \sum_{k}^{n} a_{ik} x_i x_k$, be transformed by the linear substitution, $x_i = \sum_{k}^{n} l_{ik} y_k$ $(i=1,2,\ldots,n)$, becoming $F(y_1, y_2, \ldots, y_n) \equiv \sum_{r}^{n} \sum_{s}^{n} \alpha_{rs} y_r y_s$, and denote the determinants, $\begin{vmatrix} a_{11} \ldots a_{1n} \\ \cdot \cdot \cdot \\ a_{n1} \ldots a_{nn} \end{vmatrix}$, $\begin{vmatrix} l_{11} \ldots l_{1n} \\ \cdot \cdot \cdot \\ l_{n1} \ldots l_{nn} \end{vmatrix}$, $\begin{vmatrix} \alpha_{11} \ldots \alpha_{1n} \\ \cdot \cdot \cdot \\ \alpha_{n1} \ldots \alpha_{nn} \end{vmatrix}$, respectively, by a, l, α. Then $\alpha = a l^2$.

$$F(y_1, y_2, \ldots, y_n) \equiv \sum_{i}^{n} \sum_{k}^{n} a_{ik} \left(\sum_{r}^{n} l_{ir} y_r \right) \left(\sum_{s}^{n} l_{ks} y_s \right)$$

$$\equiv \sum_{i}^{n} \sum_{k}^{n} a_{ik} \sum_{r}^{n} \sum_{s}^{n} l_{ir} l_{ks} y_r y_s$$

$$\equiv \sum_{r}^{n} \sum_{s}^{n} y_r y_s \sum_{i}^{n} \sum_{k}^{n} a_{ik} l_{ir} l_{ks}. \text{ Hence } \alpha_{rs} = \sum_{i}^{n} \sum_{k}^{n} a_{ik} l_{ir} l_{ks}.$$

The determinant of the transformed quadratic form is, accordingly:

$$\begin{vmatrix} \sum_{i} l_{i1} \sum_{k} a_{ik} l_{k1} & \sum_{i} l_{i1} \sum_{k} a_{ik} l_{k2} & \cdots & \sum_{i} l_{i1} \sum_{k} a_{ik} l_{kn} \\ \sum_{i} l_{i2} \sum_{k} a_{ik} l_{k1} & \sum_{i} l_{i2} \sum_{k} a_{ik} l_{k2} & \cdots & \sum_{i} l_{i2} \sum_{k} a_{ik} l_{kn} \\ \cdot & \cdot & \cdot & \cdot \\ \sum_{i} l_{in} \sum_{k} a_{ik} l_{k1} & \sum_{i} l_{in} \sum_{k} a_{ik} l_{k2} & \cdots & \sum_{i} l_{in} \sum_{k} a_{ik} l_{kn} \end{vmatrix}$$

This determinant is the product, columns by columns, of the determinants:

$$\begin{vmatrix} \sum_k a_{1k}l_{k1} & \sum_k a_{1k}l_{k2} & \cdots & \sum_k a_{1k}l_{kn} \\ \sum_k a_{2k}l_{k1} & \sum_k a_{2k}l_{k2} & \cdots & \sum_k a_{2k}l_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_k a_{nk}l_{k1} & \sum_k a_{nk}l_{k2} & \cdots & \sum_k a_{nk}l_{kn} \end{vmatrix} \quad \begin{vmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & \cdots & l_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{vmatrix}$$

The first of these, in turn, is the product, rows by rows, of the determinants: $\begin{vmatrix} a_{11} \cdots a_{1n} \\ \cdots \\ a_{n1} \cdots a_{nn} \end{vmatrix} \quad \begin{vmatrix} l_{11} \cdots l_{1n} \\ \cdots \\ l_{n1} \cdots l_{nn} \end{vmatrix}$.

We may add, as an immediate consequence of the form of the determinant, α, the following theorem:

Theorem 2. If $a_{ik}=a_{ki}$ $(i,k=1,2,\ldots,n)$, then $\alpha_{rs}=\alpha_{sr}$ $(r,s=1,2,\ldots,n)$

For, $$\alpha_{rs}=\sum_i\sum_k a_{ik}l_{ir}l_{ks}\equiv\sum_k\sum_i a_{ki}l_{kr}l_{is}$$

$$\alpha_{sr}=\sum_i\sum_k a_{ik}l_{is}l_{kr}\equiv\sum_k\sum_i a_{ik}l_{kr}l_{is} \quad .$$

There is, moreover, a relation between minors of the same order of the determinants, α and a, which is of the highest importance in the solution of the problem in hand. This relation was first pointed out by Sylvester*. It may be stated as an extension of Theorem 1, as follows :

Theorem 3. If the transformation is non-singular, that is, if $l\neq 0$, any minor of a is a linear function of minors of the same order of α, and conversely any minor of α is a linear function of minors of the same order of a.

*Sylvester, Phil, Mag., ser, 4, vol. 1, 1851, p. 295; Coll. Papers, vol. 1, p. 241
Bromwich, Quadratic Forms and their Classification by Means of Invariant Factors. Cambridge, 1906. p, 22.

Corresponding to an arbitrary p^{th} minor of a, obtained by deleting rows, $\theta_1, \theta_2, \ldots, \theta_p$, and columns, $\phi_1, \phi_2, \ldots, \phi_p$, consider the two following quadratic forms:

$$\alpha_{11}y_1^2 + \alpha_{12}y_1y_2 + \cdots + \alpha_{1n}y_1y_n + l_{\theta_1}y_1y_{n+1} + l_{\theta_2}y_1y_{n+2} + \cdots + l_{\theta_p}y_1y_{n+p}$$
$$+\alpha_{21}y_2y_1 + \alpha_{22}y_2^2 + \cdots + \alpha_{2n}y_2y_n + l_{\theta_1 2}y_2y_{n+1} + l_{\theta_2 2}y_2y_{n+2} + \cdots + l_{\theta_p 2}y_2y_{n+p}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$+\alpha_{n1}y_ny_1 + \alpha_{n2}y_ny_2 + \cdots + \alpha_{nn}y_n^2 + l_{\theta_1 n}y_ny_{n+1} + l_{\theta_2 n}y_ny_{n+2} + \cdots + l_{\theta_p n}y_ny_{n+p}$$
$$+l_{\phi_1}y_{n+1}y_1 + l_{\phi_2}y_{n+1}y_2 + \cdots + l_{\phi_n}y_{n+1}y_n + 0 + 0 + \cdots + 0$$
$$+l_{\phi_1}y_{n+2}y_1 + l_{\phi_2}y_{n+2}y_2 + \cdots + l_{\phi_n}y_{n+2}y_n + 0 + 0 + \cdots + 0$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$+l_{\phi_1}y_{n+p}y_1 + l_{\phi_2}y_{n+p}y_2 + \cdots + l_{\phi_n}y_{n+p}y_n + 0 + 0 + \cdots + 0 \quad ,$$

and ,

$$a_{11}x_1^2 + \cdots + a_{1\phi_1}x_1x_{\phi_1} + \cdots + a_{1\phi_p}x_1x_{\phi_p} + \cdots + a_{1n}x_1x_n + 0 + \cdots + 0$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$+a_{\theta_1 1}x_{\theta_1}x_1 + \cdots + a_{\theta_1\phi_1}x_{\theta_1}x_{\phi_1} + \cdots + a_{\theta_1\phi_p}x_{\theta_1}x_{\phi_p} + \cdots + a_{\theta_1 n}x_{\theta_1}x_n + x_{\theta_1}x_{n+1} + \cdots + 0$$
$$+a_{\theta_p 1}x_{\theta_p}x_1 + \cdots + a_{\theta_p\phi_1}x_{\theta_p}x_{\phi_1} + \cdots + a_{\theta_p\phi_p}x_{\theta_p}x_{\phi_p} + \cdots + a_{\theta_p n}x_{\theta_p}x_n + 0 + \cdots + x_{\theta_p}x_{n+p}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$+a_{n1}x_nx_1 + \cdots + a_{n\phi_1}x_nx_{\phi_1} + \cdots + a_{n\phi_p}x_nx_{\phi_p} + \cdots + a_{nn}x_n^2 + 0 + \cdots + 0$$
$$+0 + \cdots + x_{n+1}x_{\phi_1} + \cdots + 0 + \cdots + 0 + 0 + \cdots + 0$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$+0 + \cdots + 0 + \cdots + x_{n+p}x_{\phi_p} + \cdots + 0 + 0 + \cdots + 0$$

It is evident that the linear substitution,

$x_i = \sum_{\kappa}^{n} l_{i\kappa}y_\kappa$ $(i=1,2,\ldots,n)$; $x_i = y_i$ $(i=n+1, n+2, \ldots, n+p)$, will transform the second quadratic form into the first.

Denote the determinants of these two quadratic forms, and of the transformation, respectively, by α', a', and 1'. Evidently, the determinant a' is either equal to the arbitrary p^{th} minor of a originally chosen, or at most differs from it in algebraic sign. The determinant α' is nothing other than the determinant α bordered by p rows and p columns of elements from the determinant 1, and is consequently a polynomial of degree 2p in the bordering quantities; moreover, every p^{th} minor of α is the coefficient of at least one term of this polynomial*. Again 1' is either equal to 1, or at most differs from it in algebraic sign. Then, since, in accordance with Theorem 1., $\alpha' = a'1'^2$, it appears that an arbitrary p^{th} minor of a is a linear function of p^{th} minors of α.

Moreover, conversely, since $1 \neq 0$, the equations of transformation can be solved for $y_i (i=1,2,\ldots,n)$, and then by the part of the theorem already established, any p^{th} minor of α is a linear function of the p^{th} minors of a.

It should be noted that this theorem establishes the fact that the rank of a quadratic form is not changed by non-singular linear transformations.

3. PRODUCT OF DETERMINANTS. Having given two determinants,

$$\begin{vmatrix} a_{11} \cdots a_{1n} \\ \cdots \\ a_{n1} \cdots a_{nn} \end{vmatrix} \qquad \begin{vmatrix} b_{11} \cdots b_{1n} \\ \cdots \\ b_{n1} \cdots b_{nn} \end{vmatrix}$$

which we denote, respectively, by a and b, and their products,

$$\begin{vmatrix} c_{11} \cdots c_{1n} \\ \cdots \\ c_{n1} \cdots c_{nn} \end{vmatrix} \qquad \begin{vmatrix} \gamma_{11} \cdots \gamma_{1n} \\ \cdots \\ \gamma_{n1} \cdots \gamma_{nn} \end{vmatrix}$$

respectively, rows by rows, and columns by columns, which we denote by C and Γ, we wish to examine these product determinants.

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* Bocher, Higher Algebra, New York, 1907. , p.29., Th.3.

Obviously, if we choose an arbitrary minor of C, whose principal diagonal is $c_{r_1 s_1} c_{r_2 s_2} \dots c_{r_m s_m}$, this minor has for its general element, $c_{r_i s_j}$, the i^{th} row of the matrix $\left\| \begin{matrix} a_{r_1 1} & a_{r_1 2} & \dots & a_{r_1 n} \\ \dots & \dots & \dots & \dots \\ a_{r_m 1} & a_{r_m 2} & \dots & a_{r_m n} \end{matrix} \right\|$

and the j^{th} row of the matrix $\left\| \begin{matrix} b_{s_1 1} & b_{s_1 2} & \dots & b_{s_1 n} \\ \dots & \dots & \dots & \dots \\ b_{s_m 1} & b_{s_m 2} & \dots & b_{s_m n} \end{matrix} \right\|$

combined in the manner required in multiplying determinants.

We speak of the minor as the product, rows by rows, of these two matrices*. Concerning such a product, we have the following theorem:

Theorem 4. The product, rows by rows, of two matrices, with each n columns and m rows, $(m \leq n)$, is equal to the sum of the products of the minors of the m^{th} order of the one matrix respectively with the homologous minors of the other matrix.

Let the matrices be

$$\left\| \begin{matrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{matrix} \right\| \qquad \left\| \begin{matrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{matrix} \right\|$$

Then, evidently, their product, rows by rows, may be expressed as the sum of n determinants of the form:

$$\begin{vmatrix} a_{1\theta_1} b_{1\theta_1} & a_{1\theta_2} b_{2\theta_2} & \dots & a_{1\theta_m} b_{m\theta_m} \\ a_{2\theta_1} b_{1\theta_1} & a_{2\theta_2} b_{2\theta_2} & \dots & a_{2\theta_m} b_{m\theta_m} \\ \dots & \dots & \dots & \dots \\ a_{m\theta_1} b_{1\theta_1} & a_{m\theta_2} b_{2\theta_2} & \dots & a_{m\theta_m} b_{m\theta_m} \end{vmatrix} \equiv b_{1\theta_1} b_{2\theta_2} \dots b_{m\theta_m} \begin{vmatrix} a_{1\theta_1} & a_{1\theta_2} & \dots & a_{1\theta_m} \\ a_{2\theta_1} & a_{2\theta_2} & \dots & a_{2\theta_m} \\ \dots & \dots & \dots & \dots \\ a_{m\theta_1} & a_{m\theta_2} & \dots & a_{m\theta_m} \end{vmatrix}$$

where $(\theta_1, \theta_2, \dots, \theta_m)$ represents an arrangement of the integers, 1, 2, ..., n, taken m at a time, where any integer may be repeated any number of times. But we note that whenever there is repetition of an integer in a particular arrangement, the corresponding determin-

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* Pascal, Die Determinanten, Leipzig, 1900, p.27 ; p.30.

ant is identically zero. Accordingly, the sum of n^m determinants expressing the product of the matrices reduces to a sum of $\underline{m}\,\frac{n(n-1)\ldots(n-m+1)}{\underline{m}}$ determinants.

Consider first the sum of the $\underline{m}$ determinants corresponding to the $\underline{m}$ permutations of a fixed group of indices $\phi_1, \phi_2, \ldots, \phi_m$, $(\phi_1 < \phi_2 < \ldots < \phi_m)$. The absolute value of the determinant of the a's is constant for the group. It is clear, moreover, that a permutation of $\phi_1, \phi_2, \ldots, \phi_m$, which involves p inversions of order may be obtained from $\phi_1, \phi_2, \ldots, \phi_m$, by p interchanges of adjacent indices, and that this corresponds to p interchanges of columns in that determinant of the a's which has the indices $\phi_1, \phi_2, \ldots, \phi_m$, in natural order. Consequently, thesum of the $\underline{m}$ determinants corresponding to the group of indices, $\phi_1, \phi_2, \ldots, \phi_m$, may be written :

$$\begin{vmatrix} a_{1\phi_1} & \cdots & a_{1\phi_m} \\ \cdots & \cdots & \\ a_{m\phi_1} & \cdots & a_{m\phi_m} \end{vmatrix} \times \Sigma \pm b_{1\phi_1} b_{2\phi_2} \cdots b_{m\phi_m}, \text{ or, } \begin{vmatrix} a_{1\phi_1} & \cdots & a_{1\phi_m} \\ \cdots & \cdots & \\ a_{m\phi_1} & \cdots & a_{m\phi_m} \end{vmatrix} \cdot \begin{vmatrix} b_{1\phi_1} & \cdots & b_{1\phi_m} \\ \cdots & \cdots & \\ b_{m\phi_1} & \cdots & b_{m\phi_m} \end{vmatrix}$$

This is, however, the product of two homologous minors of the two given matrices. If now we consider the $\frac{n(n-1)\ldots(n-m+1)}{\underline{m}}$ possible groups of indices, we obtain the sum of the products of the homologous minors of the m^{th} order of the two matrices, as the theorem requires.

Corollary. Theorem 4 remains true if for the word rows we substitute the word columns and vice versa.

All that is necessary to adapt the demonstration of Theorem 4 to the corollary, is to interchange rows and columns in the matrices and to note that this involves merely an interchange of rows and columns in their minors of the m^{th} order. The product determinant is unchanged, and hence the demonstration of Theorem 4 applies.

We are now in a position to state a theorem of fundamental importance in the developement of the next section.

Theorem 5.* The sum of all the principal minors of order m of C is equal to the sum of all the principal minors of order m of Γ, each sum being equal to the sum of the products of all minors of order m of a, respectively with the homologous minors of b.

This theorem states, for the special case, m=n, the fact that $C=\Gamma$. In general, we note that any principal minor of C or of Γ is a product, rows by rows, if of C, columns by columns, if of , of homologous matrices of a and b. This product, in turn, is, by Theorem 4 and Corollary, equal to the sum of the products of all the minors of order m of the one matrix, respectively with the homologous minors of the other, from which fact the theorem follows at once.

4. REALITY OF THE ROOTS OF THE SECULAR EQUATION. The equation in λ,

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0, \quad \text{where} \quad a_{ik}=a_{ki},$$

is of considerable importance in geometry and mechanics. This equation has only real roots. The first simple proof of this fact, according to Pascal, was given by Sylvester.†

Let the above determinant be denoted, in functional notation, by $f(-\lambda)$. We then form the product, $f(\lambda).f(-\lambda)$, that is,

$$\begin{vmatrix} a_{11}+\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}+\lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn}+\lambda \end{vmatrix} \cdot \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

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*Pascal, Die Determinanten, p.55.

†Sylvester, Phil.Mag., ser.4, vol.4, 1852, p.138; Coll.Papers, vol.1, p.378. Pascal, p.64.

If we carry out this multiplication, rows by rows, we obtain:

$$\begin{vmatrix} c_{11}-\lambda^2 & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22}-\lambda^2 & \cdots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \cdots & c_{nn}-\lambda^2 \end{vmatrix} \quad \text{where } c_{i\kappa} = \sum_j^n a_{ij}\, a_{\kappa j} .$$

The simplicity of this product, in that λ occurs only in the elements of the principal diagonal, is an immediate consequence of the fact that $a_{i\kappa}=a_{\kappa i}$.

Let us suppose this determinant developed in ascending powers of $(-\lambda^2)$. The constant term will evidently be expressed by putting $\lambda=0$ in the above determinant; that is, it is simply the square of the determinant of the a's when we have put $\lambda=0$. Let us denote the determinant of the c's in which we have put $\lambda=0$, and the determinant of the a's in which we have put $\lambda=0$, respectively by C and A. We have then $C=A^2$ as the constant term of the equation, $f(\lambda).f(-\lambda)=0$. In order to determine the other coefficients of this equation, we imagine that to each element of the product determinant which is not a principal element, a zero is added, so that the whole determinant takes on the form of a determinant with binomial elements. This determinant can then be expressed as the sum of n^n determinants in which $(-\lambda^2)$ occurs only in columns, the remainder of whose elements are 0's. Then it is at once evident, on developing these determinants in terms of the minors of columns containing $(-\lambda^2)$, that the coefficient of $(-\lambda^2)$ will be the sum of the principal minors of order $(n-1)$ in the determinant C; that the coefficient of $(-\lambda^2)^2$ will be the sum of the principal minors of order $(n-2)$ in the determinant C, and so on; and finally, that the coefficient of $(-\lambda^2)^n$ will be unity.

But C is the product, rows by rows, of A by itself, and hence

it appears, from Theorem 5, that the equation, $f(\lambda).f(-\lambda)=0$, may be written:

$$C + C_1(-\lambda^2) + C_2(-\lambda^2)^2 + \ldots + C_p(-\lambda^2)^p + \ldots + (-\lambda^2)^n = 0,$$

where, $C = A$,

C_1 = the sum of the squares of the first minors of A,

C_2 = the sum of the squares of the second minors of A,

. . . . . .

C_p = the sum of the squares of the p^{th} minors of A.

Then, $C, C_1, C_2, \ldots, C_n=1$, are all positive numbers.

It is clear that any value of λ which satisfies the equation, $f(-\lambda)=0$, also satisfies the equation, $f(\lambda).f(-\lambda)=0$, but this latter equation cannot be satisfied by a pure imaginary of the form, $q\sqrt{-1}$, because then, $-\lambda^2=q^2$, and the introduction of this value in the equation, leads to a sum of positive numbers, which cannot be zero.

Neither can a value of λ of the complex form, $p+q\sqrt{-1}$, satisfy the equation, $f(-\lambda)=0$, because, if we put

$$a_{11} - p = a'_{11} \quad ; \quad a_{22} - p = a'_{22} \quad ,$$

the determinant

$$\begin{vmatrix} a'_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a'_{22}-\lambda & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a'_{nn}-\lambda \end{vmatrix}$$

must then be zero for $\lambda=q\sqrt{-1}$, which we have seen to be impossible.

It should be noted that here we have not only shown that the equation, $f(-\lambda)=0$, can have only real roots, but we computed the coefficients of the closely related equation, $f(\lambda).f(-\lambda)=0$.

CHAPTER II.

REDUCTION OF THE GENERAL EQUATION OF A QUADRIC TO STANDARD FORM.

5. EFFECT OF AN ORTHOGONAL SUBSTITUTION. Consider the quadratic form, $\sum_{i}^{4}\sum_{k}^{4} a_{ik}x_i x_k$ (where $a_{ik}=a_{ki}$), in which $x_4=1$. It may be written, $f(x_1,x_2,x_3)\equiv \sum_{i}^{3}\sum_{k}^{3} a_{ik}x_i x_k + 2\sum_{i}^{3} a_{i4}x_i + a_{44}$, which is the most general form of an expression of the second degree in three variables.

Suppose we transform this expression by the linear substitution, $x_i = \sum_{k}^{3} l_{ik}y_k + l_{i4}$ $(i=1,2,3)$

$x_4 = y_4$,

obtaining, $F(y_1,y_2,y_3)\equiv \sum_{r}^{3}\sum_{s}^{3}\alpha_{rs}y_r y_s + 2\sum_{r}^{3}\alpha_{r4}y_r + \alpha_{44}$. (Theorems 1 and 2). Then, $\alpha_{rs}=\sum_{i}^{4}\sum_{k}^{4} a_{ik}l_{ir}l_{ks} \equiv \sum_{i}^{3}\sum_{k}^{3} a_{ik}l_{ir}l_{ks} + l_{4r}\sum_{k}^{3} a_{4k}l_{ks} + l_{4s}\sum_{i}^{3} a_{i4}l_{ir} + a_{44}l_{4r}l_{4s}$. But $l_{41}=l_{42}=l_{43}=0$; hence, for $r=1,2,3$, $s=1,2,3$, $\alpha_{rs}=\sum_{i}^{3}\sum_{k}^{3} a_{ik}l_{ir}l_{ks}$. That is, the minor, A_{44}, is transformed precisely as if it were the determinant of the homogeneous quadratic form, $\sum_{i}^{3}\sum_{k}^{3} a_{ik}x_i x_k$, this form being subjected to the linear transformation, $x_i=\sum_{k}^{3} l_{ik}y_k$ $(i=1,2,3)$.

Hence we may state at once some important theorems. As a consequence of Theorem 1, we have:

Theorem 6. Let the non-homogeneous quadratic form, $f(x_1,x_2,x_3)\equiv \sum_{i}^{3}\sum_{k}^{3} a_{ik}x_i x_k + 2\sum_{i}^{3} a_{i4}x_i + a_{44}$, be transformed by the linear substitution, $x_i=\sum_{k}^{3} l_{ik}y_k + l_{i4}$ $(i=1,2,3)$, becoming, $F(y_1,y_2,y_3)\equiv \sum_{r}^{3}\sum_{s}^{3}\alpha_{rs}y_r y_s + 2\sum_{r}^{3}\alpha_{r4}y_r + \alpha_{44}$, and denote the determinants, $\begin{vmatrix} a_{11}\cdots a_{14} \\ \cdots \\ a_{41}\cdots a_{44}\end{vmatrix}$, $\begin{vmatrix} l_{11}l_{12}l_{13} \\ l_{21}l_{22}l_{23} \\ l_{31}l_{32}l_{33}\end{vmatrix}$, $\begin{vmatrix} \alpha_{11}\cdots\alpha_{14} \\ \cdots \\ \alpha_{41}\cdots\alpha_{44}\end{vmatrix}$, respectively, by a, l, α.

Then, $\alpha = a\,l^2$ and $\alpha_{44}=A_{44}l^2$.

And as a consequence of Theorem 3, we may add:

Theorem 7. If $1 \neq 0$, any minor of α is a linear function of minors of the same order of a, and conversely; moreover, any minor of a_{44} is a linear function of minors of A_{44} and conversely.

We may, in particular, suppose 1 to be orthogonal* and of value, +1; then, from Theorems 6 and 7, we have the following:

Theorem 8. The value and the rank of the determinant of a quadric, and the value and the rank of the determinant of its homogeneous quadratic part, are absolute invariants with regard to translations and rotations of the quadric.

6. DETERMINATION OF THE REDUCING SUBSTITUTION WHEN THE ROOTS OF THE REDUCING CUBIC ARE DISTINCT. When the non-homogeneous quadratic form, $f(x_1, x_2, x_3) \equiv \sum_{1}^{3}{}_i \sum_{1}^{3}{}_k a_{ik} x_i x_k + 2 \sum_{1}^{3}{}_i a_{i4} x_i + a_{44}$, is subjected to the linear transformation, $x_i = \sum_{1}^{3}{}_k l_{ik} y_k$ $(i=1,2,3)$,

a_{44} remains unchanged,

$2 \sum_{1}^{3}{}_i a_{i4} x_i$ gives rise to linear terms only,

$\sum_{1}^{3}{}_i \sum_{1}^{3}{}_k a_{ik} x_i x_k$ gives rise to quadratic terms only.

If, then, we can determine an orthogonal substitution, $x_i = \sum_{1}^{3}{}_k l_{ik} y_k$ $(i=1, 2, 3)$, which will reduce the form, $\sum_{1}^{3}{}_i \sum_{1}^{3}{}_k a_{ik} x_i x_k$, to a sum of positive and negative squares, we can remove the cross-products from the equation of the quadric by a pure rotation.

The determination of such an orthogonal substitution is simple enough, as we shall show, when the roots of the equation,

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0 \quad \text{are distinct.}$$

* Pascal, p. 157; Staude, Anal. Geom. des Punktes, ..., Leipzig, 1905, p. 172.

This special case of the secular equation is known as the reducing cubic, for reasons which will be at once obvious. We have seen that the roots of this equation are all real. Let us suppose that they are distinct, and denote them by $\lambda_1, \lambda_2, \lambda_3$.

Let us then set up the equations,

$$\begin{aligned}
a_{11}l_{11} + a_{12}l_{21} + a_{13}l_{31} &= \lambda_1 l_{11} \\
a_{21}l_{11} + a_{22}l_{21} + a_{23}l_{31} &= \lambda_1 l_{21} \\
a_{31}l_{11} + a_{32}l_{21} + a_{33}l_{31} &= \lambda_1 l_{31}
\end{aligned}$$

$$\begin{aligned}
a_{11}l_{12} + a_{12}l_{22} + a_{13}l_{32} &= \lambda_2 l_{12} \\
a_{21}l_{12} + a_{22}l_{22} + a_{23}l_{32} &= \lambda_2 l_{22} \\
a_{31}l_{12} + a_{32}l_{22} + a_{33}l_{32} &= \lambda_2 l_{32}
\end{aligned}$$

$$\begin{aligned}
a_{11}l_{13} + a_{12}l_{23} + a_{13}l_{33} &= \lambda_3 l_{13} \\
a_{21}l_{13} + a_{22}l_{23} + a_{23}l_{33} &= \lambda_3 l_{23} \\
a_{31}l_{13} + a_{32}l_{23} + a_{33}l_{33} &= \lambda_3 l_{33}
\end{aligned}$$

These are systems of homogeneous linear equations, with the determinant of the system, in each case, equal to zero. Hence each system has a solution other than a set of zeros. Moreover, since each system of equations determines only a proportionality among the three l's which it involves, we can so determine the l's that they will satisfy, not only the three given systems of equations, but also,

$$\begin{aligned}
l_{11}^2 + l_{21}^2 + l_{31}^2 &= 1 \\
l_{12}^2 + l_{22}^2 + l_{32}^2 &= 1 \\
l_{13}^2 + l_{23}^2 + l_{33}^2 &= 1
\end{aligned}$$

Under the particular linear transformation, the determinant of which is formed by the l's just determined,

$f(x_1, x_2, x_3) \equiv \sum_{1}^{3}{}_i \sum_{1}^{3}{}_k a_{ik} x_i x_k + 2\sum_{1}^{3}{}_i a_{i4} x_4 + a_{44}$, goes over into

$F(y_1, y_2, y_3) \equiv \sum_{1}^{3}{}_r \sum_{1}^{3}{}_s \alpha_{rs} y_r y_s + 2\sum_{1}^{3}{}_r \alpha_{r4} y_4 + a_{44}$, where

$$\alpha_{11} = \sum_{1}^{3}{}_i \sum_{1}^{3}{}_k a_{ik} l_{i1} l_{k1} \equiv \sum_{1}^{3}{}_i l_{i1} \sum_{1}^{3}{}_k a_{ik} l_{k1} = \sum_{1}^{3}{}_i l_{i1} \lambda_1 l_{i1} = \lambda_1$$

$$\alpha_{22} = \sum_{1}^{3}{}_i \sum_{1}^{3}{}_k a_{ik} l_{i2} l_{k2} \equiv \sum_{1}^{3}{}_i l_{i2} \sum_{1}^{3}{}_k a_{ik} l_{k2} = \sum_{1}^{3}{}_i l_{i2} \lambda_2 l_{i2} = \lambda_2$$

$$\alpha_{33} = \sum_{1}^{3}{}_i \sum_{1}^{3}{}_k a_{ik} l_{i3} l_{k3} \equiv \sum_{1}^{3}{}_i l_{i3} \sum_{1}^{3}{}_k a_{ik} l_{k3} = \sum_{1}^{3}{}_i l_{i3} \lambda_3 l_{i3} = \lambda_3$$

$$\alpha_{12} = \sum_{1}^{3}{}_i \sum_{1}^{3}{}_k a_{ik} l_{i1} l_{k2} \equiv \sum_{1}^{3}{}_i l_{i1} \sum_{1}^{3}{}_k a_{ik} l_{k2} = \sum_{1}^{3}{}_i l_{i1} \lambda_2 l_{i2}$$

$$\alpha_{21} = \sum_{1}^{3}{}_i \sum_{1}^{3}{}_k a_{ik} l_{i2} l_{k1} \equiv \sum_{1}^{3}{}_i l_{i2} \sum_{1}^{3}{}_k a_{ik} l_{k1} = \sum_{1}^{3}{}_i l_{i2} \lambda_1 l_{i1}$$

$$\alpha_{23} = \sum_i \sum_k a_{ik} l_{i2} l_{k3} \equiv \sum_i l_{i2} \sum_k a_{ik} l_{k3} = \sum_i l_{i2} \lambda_3 l_{i3}$$

$$\alpha_{32} = \sum_i \sum_k a_{ik} l_{i3} l_{k2} \equiv \sum_i l_{i3} \sum_k a_{ik} l_{k2} = \sum_i l_{i3} \lambda_2 l_{i2}$$

$$\alpha_{31} = \sum_i \sum_k a_{ik} l_{i3} l_{k1} \equiv \sum_i l_{i3} \sum_k a_{ik} l_{k1} = \sum_i l_{i3} \lambda_1 l_{i1}$$

$$\alpha_{13} = \sum_i \sum_k a_{ik} l_{i1} l_{k3} \equiv \sum_i l_{i1} \sum_k a_{ik} l_{k3} = \sum_i l_{i1} \lambda_3 l_{i3}$$

We have, accordingly, by Theorem 2,

$$\alpha_{12} - \alpha_{21} = (\lambda_2 - \lambda_1) \sum_i l_{i1} l_{i2} = 0$$

$$\alpha_{23} - \alpha_{32} = (\lambda_3 - \lambda_2) \sum_i l_{i2} l_{i3} = 0$$

$$\alpha_{31} - \alpha_{13} = (\lambda_1 - \lambda_3) \sum_i l_{i3} l_{i1} = 0 ,$$

whence, $\alpha_{12} = \alpha_{21} = \alpha_{23} = \alpha_{32} = \alpha_{31} = \alpha_{13} = 0$,

and since $l_{1i} l_{1j} + l_{2i} l_{2j} + l_{3i} l_{3j} = 1$ for $i=j$, $=0$ for $i \neq j$,

the determinant of the transformation is orthogonal.* (*Ref.p.11,

Whether its value is +1, or -1, depends merely on the order of the roots which we designate by $\lambda_1, \lambda_2, \lambda_3$.

7. EXISTENCE OF THE REDUCING SUBSTITUTION WHEN THE REDUCING CUBIC HAS MULTIPLE ROOTS. Let one root of the reducing cubic be λ_1. Then let $p'_{11}, p'_{21}, p'_{31}$, be determined by the equations,

$$a_{11} p'_{11} \; a_{12} p'_{21} \; a_{13} p'_{31} = \lambda_1 p'_{11}$$

$$a_{21} p'_{11} \; a_{22} p'_{21} \; a_{23} p'_{31} = \lambda_1 p'_{21}$$

$$a_{31} p'_{11} \; a_{32} p'_{21} \; a_{33} p'_{31} = \lambda_1 p'_{31}$$

If $p'_{11}, p'_{21}, p'_{31}$, are to be elements of an orthogonal determinant, the remaining six elements must be so chosen as to satisfy the following conditions:

$$p'_{11} p'_{12} \; p'_{21} p'_{22} \; p'_{31} p'_{32} = 0$$

$$p'_{11} p'_{13} \; p'_{21} p'_{23} \; p'_{31} p'_{33} = 0$$

$$p'_{12} p'_{13} \; p'_{22} p'_{23} \; p'_{32} p'_{33} = 0$$

Now $p'_{11}, p'_{21}, p'_{31}$, are not all zero. Suppose $p'_{11} \neq 0$. We then choose p'_{23}, p'_{33}, in such a way that $\begin{vmatrix} p'_{21} & p'_{31} \\ p'_{23} & p'_{33} \end{vmatrix} \neq 0$.

Then p'_{13} is determined by the second equation, and from the linearly independent* equations:

$$p'_{11}p'_{12} + p'_{21}p'_{22} + p'_{31}p'_{32} = 0$$

$$p'_{13}p'_{12} + p'_{23}p'_{22} + p'_{33}p'_{32} = 0 \quad , \quad \text{we have } \dagger$$

$$p'_{12} : p'_{22} : p'_{32} :: \begin{vmatrix} p'_{21} & p'_{31} \\ p'_{23} & p'_{33} \end{vmatrix} : - \begin{vmatrix} p'_{11} & p'_{31} \\ p'_{13} & p'_{33} \end{vmatrix} : \begin{vmatrix} p'_{11} & p'_{21} \\ p'_{13} & p'_{23} \end{vmatrix}$$

If now, we divide

$$p'_{11}, p'_{21}, p'_{31}, \text{ each by } \sqrt{p'^2_{11} + p'^2_{21} + p'^2_{31}},$$

$$p'_{12}, p'_{22}, p'_{32}, \text{ each by } \sqrt{p'^2_{12} + p'^2_{22} + p'^2_{32}},$$

$$p'_{13}, p'_{23}, p'_{33}, \text{ each by } \sqrt{p'^2_{13} + p'^2_{23} + p'^2_{33}},$$

we obtain a new set of p's satisfying the six equations we have set up, namely:

$$a_{11}p_{11} + a_{12}p_{21} + a_{13}p_{31} = \lambda_1 p_{11}$$

$$a_{21}p_{11} + a_{22}p_{21} + a_{23}p_{31} = \lambda_1 p_{21}$$

$$a_{31}p_{11} + a_{32}p_{21} + a_{33}p_{31} = \lambda_1 p_{31}$$

$$p_{11}p_{12} + p_{21}p_{22} + p_{31}p_{32} = 0$$

$$p_{11}p_{13} + p_{21}p_{23} + p_{31}p_{33} = 0$$

$$p_{12}p_{13} + p_{22}p_{23} + p_{32}p_{33} = 0$$

and moreover, satisfying also the equations:

$$p_{11}^2 + p_{21}^2 + p_{31}^2 = 1$$

$$p_{12}^2 + p_{22}^2 + p_{32}^2 = 1$$

$$p_{13}^2 + p_{23}^2 + p_{33}^2 = 1$$

Therefore the determinant of this new set of p's is orthogonal.

Let us now determine what happens to $f(x_1, x_2, x_3) \equiv \sum_i^3 \sum_k^3 a_{ik}x_i x_k + 2\sum_i^3 a_{i4}x_i + a_{44}$, under the orthogonal transformation of which the above is the determinant.

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*Bocher, Higher Algebra, p.36, ∫13, Theorem 1.

†Bocher, p.47, ∫17, Theorem 4.

As we have before noted, under such a transformation,

$a_{\mu\mu}$ remains unchanged,

$2\sum_{i}^{3} a_{i\mu}x_i$ gives rise to linear terms only,

$\sum_{i}^{3}\sum_{\kappa}^{3} a_{i\kappa}x_i x_\kappa$ gives rise to quadratic terms only.

We shall be interested, then, only in the homogeneous quadratic part of the new form. We will designate this by $\sum_{r}^{3}\sum_{s}^{3}\alpha'_{rs}y'_r y'_s$.

$$\alpha'_{11} = \sum_{i}^{3}\sum_{\kappa}^{3} a_{i\kappa}p_{i1}p_{\kappa 1} \equiv \sum_{i}^{3} p_{i1}\sum_{\kappa}^{3} a_{i\kappa}p_{\kappa 1} \equiv \sum_{i}^{3} p_{i1}\lambda_1 p_{i1} = \lambda_1 .$$

$$\alpha'_{r1} = \alpha'_{1r} = \sum_{i}^{3}\sum_{\kappa}^{3} a_{i\kappa}p_{ir}p_{\kappa 1} \equiv \sum_{i}^{3} p_{ir}\sum_{\kappa}^{3} a_{i\kappa}p_{\kappa 1} \equiv \sum_{i}^{3} p_{ir}\lambda_1 p_{i1} = 0 .$$

The determinant of the homogeneous quadratic part of the transformed form is, then,

$$\begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \alpha'_{22} & \alpha'_{23} \\ 0 & \alpha'_{32} & \alpha'_{33} \end{vmatrix}$$

It is evident that, under the same transformation, the form, $\sum_{i}^{3}\sum_{\kappa}^{3} a_{i\kappa}x_i x_\kappa - \lambda(x_1^2+x_2^2+x_3^2)$, the determinant of which is

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

goes over into a form, the determinant of which is

$$\begin{vmatrix} \lambda_1-\lambda & 0 & 0 \\ 0 & \alpha'_{22}-\lambda & \alpha'_{23} \\ 0 & \alpha'_{32} & \alpha'_{33}-\lambda \end{vmatrix}$$

By Theorem 8, these two determinants are equal for all values of λ. Therefore the two roots of the reducing cubic besides λ_1, whether distinct from λ_1 or not, are contained in the equation,

$$\begin{vmatrix} \alpha'_{22}-\lambda & \alpha'_{23} \\ \alpha'_{32} & \alpha'_{33}-\lambda \end{vmatrix} = 0$$

Let λ_2 be a root of this equation, and let q'_{22} , q'_{32} , be determined

by the equations:

$$\alpha'_{22}\, q'_{22} + \alpha'_{23} q'_{32} = \lambda_2 q'_{22}$$
$$\alpha'_{32}\, q'_{22} + \alpha'_{33} q'_{32} = \lambda_2 q'_{32}$$

If q'_{22}, q'_{32}, are to be elements of an orthogonal determinant of the second order, the other two elements must be so chosen as to satisfy the condition,

$$q'_{22}\, q'_{23} + q'_{32}\, q'_{33} = 0 .$$

Now q'_{22}, q'_{32}, are not both zero. Suppose $q'_{22} \neq 0$. We then choose $q'_{33} \neq 0$, and q'_{23} is determined by the above equation. If now, we divide

$$q'_{22}, q'_{32}, \text{ each by } \sqrt{q'^2_{22} + q'^2_{32}}$$
$$q'_{23}, q'_{33}, \text{ each by } \sqrt{q'^2_{23} + q'^2_{33}} ,$$

we obtain a new set of q's, satisfying the three equations we set up, namely:

$$\alpha'_{22}\, q_{22} + \alpha'_{23}\, q_{32} = \lambda_2 q_{22}$$
$$\alpha'_{32}\, q_{22} + \alpha'_{33}\, q_{32} = \lambda_2 q_{32}$$
$$q_{22}\, q_{23} + q_{32}\, q_{33} = 0 ,$$

and moreover, satisfying also, the equations:

$$q^2_{22} + q^2_{32} = 1$$
$$q^2_{23} + q^2_{33} = 1 .$$

Therefore the determinant of these q's is orthogonal.

Under the orthogonal transformation,

$$y_1' = y_1$$
$$y_2' = q_{22} y_2 + q_{23} y_3$$
$$y_3' = q_{32} y_2 + q_{33} y_3 ,$$

the form, $\sum_1^3 \sum_1^3 \alpha'_{rs}\, y'_r\, y'_s$, goes over into a form, the determinant of which is

$$\begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \alpha_{33} \end{vmatrix}$$

Moreover, since by reasoning employed above, the equation,

$$\alpha_{33} - \lambda = 0 ,$$

contains the third root of the reducing cubic, whether distinct from λ_1 and λ_2, or not, we have $\alpha_{33} = \lambda_3$.

Finally, since a succession of orthogonal transformations is equivalent to an orthogonal transformation whose determinant is the product of the determinants of the succession of transformations* there exists an orthogonal transformation , $x_i = \sum_{1}^{3} {}_{\kappa} l_{i\kappa} y_\kappa$ $(i=1,2,3)$, which will transform, $f(x_1, x_2, x_3) \equiv \sum_{1}^{3} {}_{i} \sum_{1}^{3} {}_{\kappa} a_{i\kappa} x_i x_\kappa + 2 \sum_{1}^{3} {}_{i} a_{i4} x_i + a_{44}$, into $F(y_1, y_2, y_3) \equiv \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + 2 \sum_{1}^{3} {}_{r} \alpha_{r4} y_r + a_{44}$. The value of the determinant of this transformation may always be made +1, for the order of the last two columns of the determinant of the p's is entirely arbitrary, and a change in the sign of the determinant of the p's will change the sign of the product.

It should be noted that we have not in any way restricted $\lambda_1, \lambda_2, \lambda_3$, to values different from zero. Evidently, in order that the original quadratic form break down into an expression in less than three variables it is necessary and sufficient that the reducing cubic have one or more zero roots.

8. DETERMINATION OF THE REDUCING SUBSTITUTION WHEN THE REDUCING CUBIC HAS MULTIPLE ROOTS. We have seen that there always exists a determinable orthogonal substitution, the determinant of which has the value +1, which will transform

$$f(x_1, x_2, x_3) \equiv \sum_{1}^{3} {}_{i} \sum_{1}^{3} {}_{\kappa} a_{i\kappa} x_i x_\kappa + 2 \sum_{1}^{3} {}_{i} a_{i4} x_i + a_{44}$$

into $$F(y_1, y_2, y_3) \equiv \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + 2 \sum_{1}^{3} {}_{r} \alpha_{r4} y_r + a_{44}$$

As a matter of fact, the same equations which determine this

* Pascal, p.159 ; Bocher, p.68.

substitution when the roots of the reducing cubic are distinct , also determine it when the reducing cubic has multiple roots, as we shall now show.

If the reducing substitution is $x_i = \sum_{1}^{3}{}_{\kappa} l_{i\kappa} y_\kappa \quad (i=1,2,3)$, we have

$$\begin{aligned} \sum_{1}^{3}{}_{i} \sum_{1}^{3}{}_{\kappa} a_{i\kappa} l_{i_1} l_{\kappa j} &= \alpha_{1j} \\ \sum_{1}^{3}{}_{i} \sum_{1}^{3}{}_{\kappa} a_{i\kappa} l_{i_2} l_{\kappa j} &= \alpha_{2j} \qquad (j=1,2,3) \\ \sum_{1}^{3}{}_{i} \sum_{1}^{3}{}_{\kappa} a_{i\kappa} l_{i_3} l_{\kappa j} &= \alpha_{3j} \end{aligned}$$

Carrying out the summation with respect to i, this is

$$\begin{aligned} l_{11}\left(\sum_{1}^{3}{}_{\kappa} a_{1\kappa} l_{\kappa j}\right) + l_{21}\left(\sum_{1}^{3}{}_{\kappa} a_{2\kappa} l_{\kappa j}\right) + l_{31}\left(\sum_{1}^{3}{}_{\kappa} a_{3\kappa} l_{\kappa j}\right) &= \alpha_{1j} \\ l_{12}\left(\sum_{1}^{3}{}_{\kappa} a_{1\kappa} l_{\kappa j}\right) + l_{22}\left(\sum_{1}^{3}{}_{\kappa} a_{2\kappa} l_{\kappa j}\right) + l_{32}\left(\sum_{1}^{3}{}_{\kappa} a_{3\kappa} l_{\kappa j}\right) &= \alpha_{2j} \\ l_{13}\left(\sum_{1}^{3}{}_{\kappa} a_{1\kappa} l_{\kappa j}\right) + l_{23}\left(\sum_{1}^{3}{}_{\kappa} a_{2\kappa} l_{\kappa j}\right) + l_{33}\left(\sum_{1}^{3}{}_{\kappa} a_{3\kappa} l_{\kappa j}\right) &= \alpha_{3j} \end{aligned}$$

This is a system of non-homogeneous linear equations, the determinant of whose coefficients has the value +1. Since $\alpha_{rs} = \lambda_r$, for r=s; =0, for r≠s, the solution is,

$$\sum_{1}^{3}{}_{\kappa} a_{1\kappa} l_{\kappa j} = \lambda_j L_{1j}; \qquad \sum_{1}^{3}{}_{\kappa} a_{2\kappa} l_{\kappa j} = \lambda_j L_{2j}; \qquad \sum_{1}^{3}{}_{\kappa} a_{3\kappa} l_{\kappa j} = \lambda_j L_{3j} .$$

Since the determinant, l, of the l's is orthogonal, $l \cdot l_{ij} = L_{ij}$ *, and

$$\sum_{1}^{3}{}_{\kappa} a_{1\kappa} l_{\kappa j} = \lambda_j l_{1j}; \qquad \sum_{1}^{3}{}_{\kappa} a_{2\kappa} l_{\kappa j} = \lambda_j l_{2j}; \qquad \sum_{1}^{3}{}_{\kappa} a_{3\kappa} l_{\kappa j} = \lambda_j l_{3j} .$$

But these are nothing other than the equations used to determine the reducing substitution when the roots of the reducing cubic are distinct.

Hence if $\lambda_1, \lambda_2, \lambda_3$, are not distinct, the set of these equations corresponding to a multiple root must have a number of linearly independent solutions equal to the multiplicity of the root. The determination of the reducing substitution, then, in any case , rests on the solution of the reducing cubic, and the subsequent solution of the nine equations of §6 (p.12), involving $\lambda_1, \lambda_2, \lambda_3$.

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* Pascal, p. 157 .

9.ELEMENTARY DIVISORS. In disscussing the solution of the set of equations corresponding to a multiple root of the reducing cubic, in the determination of the reducing substitution, it is convenient to consider certain invariants of a pair of quadratic forms.

Let A and B be two quadratic forms. Let the determinant of the family of forms,(A-λB), be denoted by $|A-\lambda B|$. If, under a non-singular linear substitution, A and B, become A' and B', it is clear from Theorem 6, that $|A-\lambda B|$ and $|A'-\lambda B'|$ can differ only by a factor which is independent of λ. Hence if $(\lambda-\lambda_1)^{k_0}$ is a factor of $|A-\lambda B|$ it is also a factor of $|A'-\lambda B'|$, and conversely.

Moreover, from Theorem 7, we see at once that if $(\lambda-\lambda_1)^{k_1}$ is a factor of all the first minors of $|A-\lambda B|$, it is also a factor of all the first minors of $|A'-\lambda B'|$, and conversely.

The same argument can be applied to minors of any order, and it follows that the sequence of indices,

$$k_0, k_1, k_2, \quad . \ . \ . \quad , k_{r-1},$$

representing respectively the highest power of the factor, $(\lambda-\lambda_1)$, contained in the determinant, $|A-\lambda B|$, in all its first minors, in all its second minors, and so on, is the same for the two determinants, $|A-\lambda B|$ and $|A'-\lambda B'|$.

That is, the factor, $(\lambda-\lambda_1)$, and the sequence of indices,

$$k_0, k_1, k_2, \quad . \ . \ . \quad , k_{r-1},$$

are invariants under any non-singular linear substitution applied to the family of forms.

The differences,

$$e_1 = k_0 - k_1, \quad e_2 = k_1 - k_2, \quad . \ . \ . \quad , e_r = k_{r-1}$$

(k_r being the first index in the sequence of k's which is zero) are known as the elementary divisors* to the base $(\lambda-\lambda_1)$ of the

*Bôcher, p.270 .

determinant of the family of forms.

Let us now consider the family of forms,

$$\sum_{1}^{3}{}_{i}\sum_{1}^{3}{}_{k} a_{ik}x_i x_k - \lambda(x_1^2+x_2^2+x_3^3) \quad ,$$

the determinant of which is

$$A(\lambda) = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

We have seen that under a certain determinable orthogonal substitution applied to the above family of forms, this determinant goes over into

$$\Delta(\lambda) = \begin{vmatrix} \lambda_1-\lambda & 0 & 0 \\ 0 & \lambda_2-\lambda & 0 \\ 0 & 0 & \lambda_3-\lambda \end{vmatrix}$$

where $\lambda_1, \lambda_2, \lambda_3$, are the roots of the reducing cubic and are not necessarily distinct.

Suppose λ_1 is a k_o-fold root of $A(\lambda) = 0$. It is also then a k_o-fold root of $\Delta(\lambda) = 0$. All principal first minors of $\Delta(\lambda)$ obtained by striking out a row and column which intersect in $(\lambda_1 - \lambda)$ will have $(\lambda - \lambda_1)^{k_o-1}$ as a factor and no higher power of $(\lambda - \lambda_1)$, while all other first minors will either vanish identically or contain $(\lambda - \lambda_1)^{k_o}$ as a factor. Similar considerations apply to the second minors. Hence the elementary divisors of $\Delta(\lambda)$, and consequently of $A(\lambda)$ are all linear.

We now note that when we substitute for λ in $A(\lambda)$, $\lambda_1, \lambda_2, \lambda_3$, we obtain respectively the determinants of the three sets of homogeneous linear equations of §6.

Suppose λ_1 is a double root of the reducing cubic, $A(\lambda) = 0$. Then $(\lambda - \lambda_1)^2$ is a factor of the determinant, $A(\lambda)$. Now from the

fact that the elementary divisors of $A(\lambda)$ are all linear, it follows that $(\lambda - \lambda_1)$ is a factor of all the first minors of $A(\lambda)$, but not of all the second minors. That is, the determinant, $A(\lambda_1)$, is of rank 3-2, and hence the system of equations corresponding to λ_1 has two linearly independent solutions.*

Similar considerations apply in case λ_1 is a 3-fold root.

We may now generalize the statement at the close of §6, as follows:

Whether the value of the determinant of the reducing substitution is +1 or -1, depends merely on the order of the linearly independent solutions which we designate as corresponding to $\lambda_1, \lambda_2, \lambda_3$, these roots not being necessarily distinct.

10. FINAL SIMPLIFICATION. Having reduced the form, $\sum_{i}^{3}\sum_{k}^{3} a_{ik} x_i x_k + 2\sum_{i}^{3} a_{i4} x_i + a_{44}$, to the form, $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + 2\alpha_{14} y_1 + 2\alpha_{24} y_2 + 2\alpha_{34} y_3 + a_{44}$, we can effect a still further simplification, provided $\lambda_1, \lambda_2, \lambda_3$, are not all zero.

If we subject this latter form to the translation,

$y_1 = z_1 + l_{14}, \quad y_2 = z_2 + l_{24}, \quad y_3 = z_3 + l_{34}$, we obtain,

$$\lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2 + 2(\lambda_1 l_{14} + \alpha_{14}) z_1 + 2(\lambda_2 l_{24} + \alpha_{24}) z_2 + 2(\lambda_3 l_{34} + \alpha_{34}) z_3 + 2\sum_{r}^{3} \alpha_{r4} l_{r4} + \sum_{r}^{3} \lambda_r l_{r4}^2 + a_{44}$$

Hence, if $\lambda_i \neq 0$ we can so fix l_{i4} that the coefficient of z_i is zero. We have, then, three cases:

Case I.	$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \neq 0$	Reduced form, $\lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2 + c_{44}$
Case II.	$\lambda_1 \lambda_2 \neq 0, \lambda_3 = 0$	Reduced form, $\lambda_1 z_1^2 + \lambda_2 z_2^2 + 2c'_{34} z_3 + c'_{44}$
Case III.	$\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 = 0$	Reduced form, $\lambda_1 z_1^2 + 2b_{24} z_2 + 2b_{34} z_3 + c'_{44}$

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* Bôcher, Introd. to Higher Algebra, p.50, Th.1,

The third form admits of still further simplification when $b_{24} \neq 0$, for, if we subject it to the orthogonal transformation,

$$z_1' = z_1 \quad , \quad z_2' = \frac{b_{34}}{\sqrt{b_{34}^2 + b_{24}^2}} z_2 + \frac{-b_{24}}{\sqrt{b_{34}^2 + b_{24}^2}} z_3, \quad z_3' = \frac{b_{24}}{\sqrt{b_{34}^2 + b_{24}^2}} z_2 + \frac{b_{34}}{\sqrt{b_{34}^2 + b_{24}^2}} z_3$$

we have $2b_{24} z_2 + 2b_{34} z_3 = 2\sqrt{b_{34}^2 + b_{24}^2}\ z_3'$.

Under this transformation, $\lambda_1 z_1^2$ goes into $\lambda_1 z_1'^2$ and c_{44} remains c_{44}.

We may now write as the reduced forms:

Case I.	$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \neq 0$	Reduced form, $\lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2 + c_{44}$
Case II.	$\lambda_1 \cdot \lambda_2 \neq 0, \lambda_3 = 0$	Reduced form, $\lambda_1 z_1^2 + \lambda_2 z_2^2 + 2c'_{34} z_3 + c'_{44}$
Case III.	$\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 = 0$	Reduced form, $\lambda_1 z_1^2 + 2c'_{34} z_3 + c'_{44}$

It is, moreover, always possible, in case c'_{34} and c'_{44} are both different from zero, to cause the constant term to vanish by a simple translation. For, putting $z_3 = \zeta_3 + \gamma$, we have for the part of the form affected, $c_{44} = 2c'_{34}\gamma + c'_{44}$, and hence we can always so choose γ as to make $c_{44} = 0$.

Since it is easily verified that a translation followed by a rotation can always be replaced by the same rotation followed by a determinable translation, it is plain that the general equation of a quadric may always be reduced to its most simple form by a single rotation (about the origin) followed by a single translation.

The final reduced form is then :

$$\lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2 + 2c_{34} z_3 + c_{44} \quad ,$$

where either λ_3 or c_{34} is zero, and either c_{34} or c_{44} is zero.

CHAPTER III.

CLASSIFICATION OF THE REDUCED FORMS.

11. DIRECT DETERMINATION OF THE COEFFICIENTS OF THE REDUCED FORM. Under the rotation and translation which reduces the general equation of a quadric to standard form, the equation $\sum_{1}^{3}{}_{i}\sum_{1}^{4}{}_{k} a_{ik}x_i x_k + 2\sum_{1}^{3}{}_{i} a_{i4}x_i + a_{44} - \lambda(x_1^2 + x_2^2 + x_3^2) = 0$, goes over into $\lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2 + 2c_{34}z_3 + c_{44} - \lambda\left\{(z_1+l_{14})^2 + (z_2+l_{24})^2 + (z_3+l_{34})^2\right\} = 0$.

Hence, by Theorem 8,

$$a_0(\lambda) = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22}-\lambda & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33}-\lambda & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

is equal to

$$a_1(\lambda) = \begin{vmatrix} \lambda_1-\lambda & 0 & 0 & -\lambda l_{14} \\ 0 & \lambda_2-\lambda & 0 & -\lambda l_{24} \\ 0 & 0 & \lambda_3-\lambda & c_{34}-\lambda l_{34} \\ -\lambda l_{14} & -\lambda l_{24} & c_{34}-\lambda l_{34} & c_{44}-\lambda\sum_{1}^{3}{}_{i} l_{i4}^2 \end{vmatrix}$$

for all values of λ .

Similarly for the principal first minors in the upper left-hand corners, which we denote respectively by $A_0(\lambda)$, and $A_1(\lambda)$.

Expanding $a_1(\lambda)$ by Cauchy's expansion,* we obtain

$$a_1(\lambda) = (c_{44}-\lambda l_{14}^2-\lambda l_{24}^2-\lambda l_{34}^2)A_1(\lambda) - \lambda^2 l_{14}^2(\lambda_2-\lambda)(\lambda_3-\lambda) - \lambda^2 l_{24}^2(\lambda_1-\lambda)(\lambda_3-\lambda) - (c_{34}-\lambda l_{34})^2(\lambda_1-\lambda)(\lambda_2-\lambda)$$

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* Mathews' revision of Scott's Determinants (Cambridge, 1904), p.47.

Pascal, Die Determinanten, p.43.

Hence,

$$\frac{a_1(\lambda)}{A_1(\lambda)} \equiv \frac{a_0(\lambda)}{A_0(\lambda)} \equiv c_{44} + \frac{\begin{array}{l}\{-\lambda_1\lambda_2 c_{34}^2\} \\ +\{-\lambda_1\lambda_2\lambda_3(l_{14}^2+l_{24}^2+l_{34}^2)+(\lambda_1+\lambda_2)c_{34}^2+2l_{34}\lambda_1\lambda_2 c_{34}\}\ \lambda \\ +\{+\lambda_2\lambda_3(l_{24}^2+l_{34}^2)+\lambda_3\lambda_1(l_{34}^2+l_{14}^2)+\lambda_1\lambda_2(l_{14}^2+l_{24}^2)-2l_{34}(\lambda_1+\lambda_2)c_{34}-c_{34}^2\}\lambda^2 \\ +\{-(\lambda_1 l_{14}^2+\lambda_2 l_{24}^2+\lambda_3 l_{34}^2)+2l_{34}c_{34}\}\ \lambda^3\end{array}}{\begin{array}{l}\{+\lambda_1\lambda_2\lambda_3\} \\ +\{-(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2)\}\lambda \\ +\{+\lambda_1+\lambda_2+\lambda_3\}\ \lambda^2 \\ +\{-1\}\ \lambda^3\end{array}}$$

We shall now show that we can determine the coefficients of the reduced form directly from the original equation of the quadric without having to determine the reducing substitution.

From the above identity, the following is obvious:

RULE

First, factorize $A_0(\lambda)$, obtaining the real numbers $\lambda_1, \lambda_2, \lambda_3$.

Second, determine the first terms of $a_0(\lambda)$ and $A_0(\lambda)$, respectively, when the terms of these polynomials are arranged in ascending powers of λ.

Finally, according as the quadric is of type I., II., or III., (below) the power of λ in the first term of $a_0(\lambda)$ is the same as, higher than, or lower than the power of λ in the first term of $A_0(\lambda)$.

TYPE I. $c_{34}=0$, $c_{44}\neq 0$. First term of $\frac{a_0(\lambda)}{A_0(\lambda)}$ is c_{44}.

TYPE II. $c_{34}=0$, $c_{44}=0$. First term of $\frac{a_0(\lambda)}{A_0(\lambda)}$ contains positive power

TYPE III. $c_{34}\neq 0$, $c_{44}=0$. First term of $\frac{a_0(\lambda)}{A_0(\lambda)}$ is $\frac{c_{34}^2}{\lambda}$.

Having thus determined, λ_1, λ_2, λ_3, the type, and c_{44} if the type is I., c_{34} if the type is III, the reduced equation may be written at once, as follows:

FIRST TYPE	Reduced equation,	$\lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2 + c_{44} = 0$	
I,	$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \neq 0$		Central Quadric
I_2	$\lambda_1 \cdot \lambda_2 \neq 0$, $\lambda_3 = 0$		Elliptic or Hyperbolic Cylinder,
I_3	$\lambda_1 \neq 0$, $\lambda_2 = 0$, $\lambda_3 = 0$		Two Parallel Planes
SECOND TYPE	Reduced equation,	$\lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2 = 0$	
II,	$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \neq 0$		Cone
II_2	$\lambda_1 \cdot \lambda_2 \neq 0$, $\lambda_3 = 0$		Two Intersecting Planes
II_3	$\lambda_1 \neq 0$, $\lambda_2 = 0$, $\lambda_3 = 0$		Two Coincident Planes
THIRD TYPE	Reduced equation	$\lambda_1 z_1^2 + \lambda_2 z_2^2 + 2c_{34} z_3 = 0$	
III_2	$\lambda_1 \cdot \lambda_2 \neq 0$, $\lambda_3 = 0$		Paraboloid
III_3	$\lambda_1 \neq 0$, $\lambda_2 = 0$, $\lambda_3 = 0$		Parabolic Cylinder.

Similarly, for the conics, we obtain:

FIRST TYPE	Reduced equation,	$\lambda_1 z_1^2 + \lambda_2 z_2^2 + c_{33} = 0$	
I,	$\lambda_1 \cdot \lambda_2 \neq 0$		Ellipse or Hyperbola
I_2	$\lambda_1 \neq 0$, $\lambda_2 = 0$		Two Parallel Lines
SECOND TYPE	Reduced equation,	$\lambda_1 z_1^2 + \lambda_2 z_2^2 = 0$	
II,	$\lambda_1 \cdot \lambda_2 \neq 0$		Two Intersecting Lines
II_2	$\lambda_1 \neq 0$, $\lambda_2 = 0$		Two Coincident Lines
THIRD TYPE	Reduced equation	$\lambda_1 z_1^2 + 2c_{23} z_2 = 0$	
III_2	$\lambda_1 \neq 0$, $\lambda_2 = 0$		Parabola

12. EXAMPLES. As an illustration of the general method of the last article, consider the quadric,

$4x^2 + y^2 + 4z^2 - 4yz + 8zx - 4xy + 2x - 4y + 5z + 1 = 0$ [*1]

$$a_0(\lambda) = \begin{vmatrix} 4-\lambda & -2 & 4 & 1 \\ -2 & 1-\lambda & -2 & -2 \\ 4 & -2 & 4-\lambda & 5/2 \\ 1 & -2 & 5/2 & 1 \end{vmatrix} = \frac{81}{4}\lambda + \cdots\cdots$$

$$A_0(\lambda) = 9\lambda^2 - \lambda^3 \qquad \lambda_1 = 9\,,\ \lambda_2 = 0\,,\ \lambda_3 = 0\,.$$

Then, $\frac{a_0(\lambda)}{A_0(\lambda)} = \frac{9/4}{\lambda}$ ..... The quadric is therefore of Type III., and since we have $2c_{34}=3$, the reduced equation is, $9z_1^2 + 3z_3 = 0$, or, $3z_1^2 + z_3 = 0$. Hence the latus-rectum of a principal parabolic section is 1/3.

Again, consider the quadric,

$11x^2 + 10y^2 + 6z^2 - 8yz + 4zx - 12xy - 12 = 0$ [*2]

$$a_0(\lambda) = \begin{vmatrix} 11-\lambda & -6 & 2 & 0 \\ -6 & 10-\lambda & -4 & 0 \\ 2 & -4 & 6-\lambda & 0 \\ 0 & 0 & 0 & -12 \end{vmatrix}$$

$$= -12(324 - 180\lambda + 27\lambda^2 - \lambda^3)$$

$$A_0(\lambda) = 324 - 180\lambda + 27\lambda^2 - \lambda^3 \qquad \lambda_1 = 3\,,\ \lambda_2 = 6\,,\ \lambda_3 = 18\,.$$

The quadric is evidently of Type I., with $c_{4.4} = -12$. The reduced equation is therefore,

$3z_1^2 + 6z_2^2 + 18z_3^2 - 12 = 0$, or, $z_1^2 + 3z_2^2 + 6z_3^2 - 4 = 0$.

[*1] Discussed by the ordinary method in C. Smith, Solid Geometry, p.63.

[*2] Salmon, Geometry of Three Dimensions, p.71.

CHAPTER IV.

CLASSIFICATION BY MEANS OF RANK .

13. DIRECT DETERMINATION OF THE TYPE OF CONIC OR QUADRIC. It is quite often desirable to know merely the type of a conic or quadric, any further knowledge not being essential.

Such a determination can be made directly, as we shall now show, without any further calculation than the determination of the rank of the determinant of the equation, and the rank of its first principal minor in the upper left-hand corner.

By Theorem 8, Chap.II., the rank of each of these determinants is an absolute invariant under rigid displacements of the conic or quadric; that is, these ranks will be the same for the equation in its simplest reduced form as for the original unreduced equation.

If, then, we determine the rank of the determinant of the general equation,

$$\lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2 + 2c_{34} z + c_{44} = 0 \quad ,$$

and the rank of its principal first minor in the upper left-hand corner, for each of the cases of p.25, which cases represent merely the various possibilities as to the vanishing of the coefficients, $\lambda_1, \lambda_2, \lambda_3, c_{34}, c_{44}$, these ranks will be characteristic of the type of quadric in each case, in whatever form its equation may be.

Now it fortunately happens that these ranks are distinct for the several types of quadric, and moreover all the possibilities as to rank are exhausted by these several types.

Hence we have the following convenient means of classification by a mere inspection of the equation in whatever form it may happen to be.

Rank of the determinant of the original unreduced equation.	Rank of its principal first minor in the upper left-hand corner.	TYPE
4	3	I_4 Central Quadric
4	2	III_2 Paraboloid
3	3	II_4 Cone
3	2	I_2 Elliptic or Hyperbolic Cyl.
3	1	III_3 Parabolic Cylinder
2	2	II_2 Two Intersecting Planes
2	1	I_3 Two Parallel Planes
1	1	II_3 Two Coincident Planes
1	0	Equation linear

The necessary changes in the case of conics will be recognized without difficulty. We obtain:

3	2	I_4 Ellipse or Hyperbola
3	1	III_2 Parabola
2	2	II_4 Two Intersecting Lines
2	1	I_2 Two Parallel Lines
1	1	II_2 Two Coincident Lines
1	0	Equation linear

14. As an example of the foregoing method, consider the quadric *,

$$4x^2 + y^2 + 4z^2 - 4yz + 8zx - 4xy + 8x - 4y + 5z + 1 = 0 .$$

The determinant to be considered is,

$$\begin{vmatrix} 4 & -2 & 4 & 1 \\ -2 & 1 & -2 & -2 \\ 4 & -2 & 4 & 5/2 \\ 1 & -2 & 5/2 & 1 \end{vmatrix}$$

Adding twice the second column to the first and to the third, which does not change the rank of either of the determinants with which we are concerned *, we have,

$$\begin{vmatrix} 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & -2 & 0 & 5/2 \\ -3 & -2 & -3/2 & 1 \end{vmatrix}$$

whence it is evident that the determinant of the original equation is of rank 3, and that its principal first minor in the upper left-hand corner is of rank 1.

The quadric represented is consequently a parabolic cylinder.

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* C. Smith, Solid Geometry, p.63.

* Bôcher, Introduction to Higher Algebra, p.55.

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